Pro-C n-crossed modules

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Abstruct:

In this paper we introduce and study a new concept in the theory of crossed modules which we call "n-crossed module", and we define the morphisms between n-crossed modules. Then we give the Pro-C analoges for these concepts with several various results on constructing Pro-C n-crossed modules from a given Pro-C n-crossed modules. Finally we give and study the pull-back concept in the category of Pro-C n-crossed modules.

الخلاصة

يتضمن هذا البحث تقديم ودراسة مفهوم جديد في نظرية الموديولات المتصالبة و الذي نطلق عليه " موديول n - متصالب " , كما عرفنا الأسهم بين الموديولات n - متصالبة . ثم أدخلنا بنية تبولوجية C - اسقاطية لهذه المفاهيم مع العديد من النتائج حول كيفية بناء موديولات n - متصالبة C - اسقاطية من موديولات n - متصالبة C - اسقاطية معطاة. و اخيرا" درسنا مفهوم السحب الخلفي في فصيلة الموديولات n - متصالبة C - اسقاطية .

Introduction:

Crossed module are usefully regaded as 2-dimensional forms of groups, they introduced by J.H.C. Whitehead in [10].Crossed modules occur in the theory of group presentations,in group cohomology and in providing algebraic modules for certain homotopy types, for history we refer to [7],[8] and [11] .

There are profinite analogues of each of these contexts. A profinite group is a projective (inverse) limit of projective (inverse) system of finite groups, where the finite groups are given the discrete topology. Moreover, the profinite group is a compact,Hausdorff and totally disconnected topological group since the open normal subgroups of such group serves as a neighbourhood basis of the identity. For basic definitions and results in the theory of profinite group we refer to [3] and [9].

In this paper C will denote a class of finite groups which closed under the formation of subgroups, homomorphic images,qoutient groups and finite products. Pro-C groups are profinite groups whose finite quotients are in C. Hence Pro-C group is a natural generalization of profinite group.

Almost all of the algebraic results of crossed modules would seem to generalise, with suitable modifications, to the case where the groups involved are profinite groups or Pro-C groups and the homomorphism and action are continuous. For previous work we refer to [4] and [5].For recent work see [1].

We recall here that a crossed G-module (B,G;δ) is a group homomorphism $\delta: B \to G$ and an action of a group G on the left of a group B, $(g,b) \mapsto g.b = {}^{g}b$,such that satisfies the following two axioms:

(CM1) $\delta(g.b)=g\delta(b)g^{-1}$ for all $g \in G$, $b \in B$;

(CM2) $\delta(b_2).b_1= b_2\, b_1\, b_2^{-1}$, for all $b_1,b_2 \in B$.

The usual notation of a crossed module $\delta:B\to G$ is $(B,G;\delta)$ regardless of whether the action of G on B is from left or right. In this paper we need to distinguish between the sides of that action. So, if the action of G on B is from left ,we will write the crossed module $\delta:B\to G$ as $(\delta,G;B)$, i.e. by putting G to the left of B,and if the action of G on B is from right ,we will write the crossed module $\delta:B\to G$ as $(B;\delta,G)$,i.e. by butting G to the right of B. This new notation makes our study to this new concept easy. For example if $(\delta_1,G_1;B)$, $(\delta_2,G_2;B)$,...., $(\delta_n,G_n;B)$ are left crossed modules ,then we can simply use the notation $(\delta_1,G_1;\ \delta_2,G_2\ ;\ldots;\delta_n,G_n\ ;B)$ to denotes a left n-crossed module. Similarly the notation $(B;\delta_1,G_1;\ \delta_2,G_2\ ;\ldots;\delta_n,G_n)$ denotes a right n-crossed module.

We recall here that a morphism (θ_1,θ_2): $(\delta,G;B)\to(\delta',G';B')$ of left crossed module consists of group homomorphisms $\theta_1:G\to G'$, $\theta_2:B\to B'$ such that , $\delta'\theta_2=\theta_1\delta$ and $\theta_2({}^{g}b)={}^{\theta_1(g)}\theta_2(b)$ for all $g\in G$ and $b\in B$.

1- n-Crossed Module:

Definition(1-1):

A left n-crossed module $(\delta_1,G_1;\ \delta_2,G_2\ ;\ldots;\delta_n,G_n\ ;B)$ consists of left crossed modules $(\delta_1,G_1;B)$, $(\delta_2,G_2;B)$,...., $(\delta_n,G_n;B)$ such that:

$$g_n.(g_{n-1}.(\ldots.(g_2.(g_1.b))\ldots)=g_{i_n}.(g_{i_{n-1}}.(\ldots.(g_{i_2}.(g_{i_1}.b))\ldots)\quad\ldots(1\text{-}1\text{-}1)$$

for all $g_1\in G_1$, $g_2\in G_2,\ldots,g_n\in G_n$ and $b\in B$,where $i_1\neq i_2\neq\ldots\neq i_n=1,2,\ldots,n$; and $i_j\neq j$ for any $j=1,2,\ldots,n$.

The condition (1-1-1) means that the left n-actions of the groups $G_1,G_2,\ldots,G_n$ on B are compatible, and that the permutation of the left n-actions of the groups $G_1,G_2,\ldots,G_n$ on B which are commute with each other is equal to n! .

To explain the definition (1-1) and the condition (1-1-1), suppose n=2 , then we have a left 2-crossed module or bi-crossed module,$(\delta_1,G_1;\delta_2,G_2;B)$,{for basic definition we refer to [6] },such that;

$$g_2.(g_1.b)=g_1.(g_2.b)$$

for all $g_1\in G_1$, $g_2\in G_2$ and $b\in B$. We see here that the permutation of the two left actions of G_1 and G_2 on B which are commutative with each other is equal to 2!.

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Also ,if we suppose that n=3 , then we have a left 3-crossed module $(\delta_1,G_1;\delta_2,G_2;\delta_3,G_3;B)$ such that ;

$$g_3.(g_2.(g_1.b))=g_3.(g_1.(g_2.b))=g_1.(g_3.(g_2.b))=g_1.(g_2.(g_3.b))=g_2.(g_1.(g_3.b))=g_2.(g_3.(g_1.b))$$

for all $g_1\in G_1$, $g_2\in G_2$, $g_3\in G_3$ and $b\in B$. We see here that the permutation of the 3-actions of $G_1,G_2$and G_3 on B which are commutative with each other is equal to 3!=6 .

Definition (1-2):

Let $(\delta_1,G_1;\delta_2,G_2;\ldots;\delta_n,G_n;B)$ and $(\sigma_1,H_1;\sigma_2,H_2;\ldots;\sigma_n,H_n;M)$ be left n-crossed modules .An n-morphism ;
$(\theta_1,\theta_2,\ldots,\theta_{n+1}):(\delta_1,G_1;\delta_2,G_2;..;\delta_n,G_n;B)\to(\sigma_1,H_1;\sigma_2,H_2;..;\sigma_n,H_n;M)$ Consists of morphisms $(\theta_1,\theta_{n+1}):(\delta_1,G_1;B)\to(\sigma_1,H_1;M)$, $(\theta_2,\theta_{n+1}):(\delta_2,G_2;B)\to(\sigma_2,H_2;M)$,…,$(\theta_n,\theta_{n+1}):(\delta_n,G_n;B)\to(\sigma_n,H_n;M)$ of left crossed modules .

We remark here the n-morphism $(\theta_1,\theta_2,\ldots,\theta_{n+1})$ preserves the compatibility of the n left actions of the groups $G_1,G_2,\ldots,G_n$ on B , i.e.
$\theta_1(g_1).(\theta_2(g_2).(\ldots.(\theta_n(g_n).\theta_{n+1}(b))\ldots)=\theta_{i_1}(g_{i_1}).(\theta_{i_2}(g_{i_2}).(\ldots.(\theta_{i_n}(g_{i_n}).\theta_{n+1}(b))\ldots)$
where $i_1\neq i_2\neq\ldots\neq i_n=1,2,\ldots,n$.

The Pro-C analogues of definitions (1-1) and (1-2) are now easy to give.

Definition (1-3):

A left Pro-C n-crossed module $(\delta_1,G_1;\delta_2,G_2;\ldots;\delta_n,G_n;B)$ is a left n- crossed module in which $G_1,G_2,\ldots,G_n$ and B are Pro-C topological groups ,each of $G_1,G_2,\ldots,G_n$ acts continuously on the left of B and the group homomorphisms $\delta_1:B\to G_1$,$\delta_2:B\to G_2,\ldots,\delta_n:B\to G_n$ are continuous.

Definition (1-4):

A (contiuous) n-morphism

$$(\theta_1,\theta_2,\ldots,\theta_{n+1}):(\delta_1,G_1;\delta_2,G_2;\ldots;\delta_n,G_n;B)\to(\sigma_1,H_1;\sigma_2,H_2;\ldots;\sigma_n,H_n;M)$$

of left Pro-C n-crossed modules is a n-morphism between left n-crossed modules in which $\theta_1:G_1\to H_1$, $\theta_2:G_2\to H_2,\ldots,\theta_n:G_n\to H_n$ and $\theta_{n+1}:B\to M$ are continuous group homomorphisms.

Proposition (1-5):

Let a Pro-C group G acts continuously from the right on a Pro-C group B. For all $g\in G$ and $b\in B$, define

$$^{g}b = b^{g^{-1}} \qquad \ldots(1\text{-}5\text{-}1)$$

which is a continuous left action of G on B , and that $(B;\delta,G)$ is a right Pro-C crossed module if, and only if, $(\delta,G;B)$ is a left Pro-C crossed module.

Proof:
It easy to check that the relation (1-5-1) defines a continuous left action of a Pro-C group G on a Pro-C group B.

Now suppose that $(B;\delta,G)$ is a right Pro-C crossed module.Therefore $\delta:B\rightarrow G$ is a continuous group homomorphism .So, we need only to satisfy the crossed module axioms (CM1) and (CM2) :

(CM1) for all $g\in G$ and $b\in B$,

$$\delta({}^{g}b) = \delta(b^{g^{-1}})$$

$$= (g^{-1})^{-1}\,\delta(b)\,g^{-1} \text{ , (since } (B;\delta,G) \text{ is a right Pro-C crossed module)}$$

$$= g\,\delta(b)\,g^{-1}$$

(CM2) for all $b_1,b_2 \in B$,

$${}^{\delta(b_2)}b_1 = b_1^{(\delta(b_2))^{-1}}$$

$$= (b_2^{-1})^{-1} b_1\, b_2^{-1} \text{ , (since } (B;\delta,G) \text{ is a right Pro-C crossed module)}$$

$$= b_2\, b_1\, b_2^{-1}$$

Thus $(\delta,G;B)$ is a left Pro-C crossed module. Similarly, we can prove the converse direction.

1- Various results on Pro-C n-crossed module :

In this section we give a several various results on constructing Pro-C n-crossed modules from a given Pro-C n-crossed module or a given Pro-C crossed module.

Proposition (2-1):

The following statements are equivalent :

(i) $(\delta_1,G_1;\delta_2,G_2;\ldots;\delta_n,G_n;B)$ is a left Pro-C n-crossed module.

(ii) $(B;\delta_1,G_1;\delta_2,G_2;\ldots;\delta_n,G_n)$is a right Pro-C n-crossed module.

(iii) $(\delta_1,G_1;\delta_2,G_2;\ldots;\delta_i,G_i;B;\delta_{i+1},G_{i+1};\ldots;\delta_n,G_n)$ is a Pro-C n-crossed module which is left from 1 to i and is right from i+1 to n .

Proof:

(i)→(ii): Suppose (i), therefore $(\delta_1,G_1;B),(\delta_2,G_2;B),\ldots,(\delta_n,G_n;B)$ are left Pro-C n-crossed modules.From proposition (1-5) there are right Pro-C crossed modules $(B;\delta_1,G_1), (B;\delta_2,G_2),\ldots, (B;\delta_n,G_n)$ such that;

$$b^{g_i} = {}^{g_i^{-1}}b \quad \ldots\ldots(2\text{-}1\text{-}1)$$

for all $b \in B$, $g_i \in G_i$ and $i=1,2,\dots,n$. To prove (ii) we need only to show that ;

$$(\dots((b.g_1).g_2)\dots .g_{n-1}).g_n = (\dots((b.g_{i_1}).g_{i_2})\dots .g_{i_{n-1}}).g_{i_n}$$

for all $g_1 \in G_1, g_2 \in G_2, \dots, g_n \in G_n$ and $b \in B$, where $i_1 \neq i_2 \neq \dots \neq i_n = 1,2,\dots,n$.

From (2-1-1) we have ;

$$(\dots((b.g_1).g_2)\dots .g_{n-1}).g_n = g_n^{-1}.(g_{n-1}^{-1}.(\dots .(g_2^{-1}.(g_1^{-1}.b))\dots)$$

$$= g_{i_n}^{-1}.(g_{i_{n-1}}^{-1}.(\dots.(g_{i_2}^{-1}.(g_{i_1}^{-1}.b))\dots) \quad \text{, (since}$$

$(\delta_1,G_1;\delta_2,G_2;\dots;\delta_n,G_n;B)$ is a left Pro-C n-crossed module)

$$= (\dots((b.g_{i_1}).g_{i_2})\dots .g_{i_{n-1}}).g_{i_n} .$$

Similarly , we can prove (ii) → (iii) and (iii) → (i) .

Proposition (2-2):

If $(\delta,G;B)$ is a left Pro-C crossed module, then $(B;\delta,G;\delta,G;\dots;\delta,G)$ is a right Pro-C n-crossed module provided that G is abelian.

Proof:

Since $(\delta,G;B)$ is a left Pro-C crossed module, then by proposition (1-5) we have $(B;\delta,G)$ is a right Pro-C crossed module with

$$b^g = {}^{g^{-1}}b \quad \dots\dots\dots(2\text{-}2\text{-}1)$$

for all $g \in G$ and $b \in B$. We need only to show that

$$(\dots((b.g_1).g_2)\dots .g_{n-1}).g_n = (\dots((b.g_{i_1}).g_{i_2})\dots .g_{i_{n-1}}).g_{i_n}$$

for all $g_1,g_2,\dots,g_n \in G$ and $b \in B$, where $i_1 \neq i_2 \neq \dots \neq i_n = 1,2,\dots,n$.

From (2-2-1) we have;

$$(\dots((b.g_1).g_2)\dots .g_{n-1}).g_n = g_n^{-1}.(g_{n-1}^{-1}.(\dots .(g_2^{-1}.(g_1^{-1}.b))\dots)$$

$$= (g_n^{-1}g_{n-1}^{-1}\dots g_2^{-1}g_1^{-1}).b$$

$$= (g_{i_n}^{-1}g_{i_{n-1}}^{-1}\dots g_{i_2}^{-1}.g_{i_1}^{-1}).b$$

$, i_1 \neq i_2 \neq \dots \neq i_n = 1,2,\dots,n$.(since G is abelian)

$$= g_{i_n}{}^{-1}.(g_{i_{n-1}}{}^{-1}.(\ldots(g_{i_2}{}^{-1}.(.g_{i_1}{}^{-1}.b))\ldots)$$

$$= (\ldots((b.\,g_{i_1}).\,g_{i_2})\ldots\,.\,g_{i_{n-1}}).\,g_{i_n}$$

Similarly, we can show that if $(\delta,G;B)$ is a left Pro-C crossed module then $(\delta,G;\delta,G;\ldots;\delta,G;B)$ is a left Pro-C n-crossed module and $(\delta,G;\delta,G;\ldots;\delta,G;B;\delta,G;\ldots;\delta,G)$ is a left-right Pro-C n-crossed module provided that G is abelian Pro-C group.

Proposition (2-3):

If $(\delta_1,G_1;\delta_2,G_2;\ldots;\delta_n,G_n;B)$ is a left Pro-C n-crossed module, then :

(i) $(\alpha_1,G_1;\alpha_2,G_2;\ldots;\alpha_n,G_n;B\times B)$ is a left Pro-C n-crossed module,provided that B is abelian.

(ii) $(\alpha_1,G_1;\alpha_2,G_2;\ldots;\alpha_i,G_i;\eta_{i+1},G_{i+1}\times G_{i+1};\eta_{i+2},G_{i+2}\times G_{i+2};\ldots;\eta_n,G_n\times G_n;B\times B)$ is a left Pro-C n-crossed module.

(iii) $(\eta_1,G_1\times G_1;\eta_2,G_2\times G_2;\ldots;\eta_n,G_n\times G_n;B\times B)$ is a left Pro-C n-crossed module.

Proof:

To prove (i) we need to show the following :

(i-1) $(\alpha_i,G_i;B\times B)$ is a left Pro-C crossed module.

(i-2) For all $g_1\in G_1$, $g_2\in G_2,\ldots, g_n\in G_n$ and m , $b\in B$;

$g_n.(g_{n-1}.(\ldots(g_2.(g_1.(m,b)))\ldots) = g_{i_n}.(g_{i_{n-1}}.(\ldots(g_{i_2}.(g_{i_1}.(m,b)))\ldots)$

where $i_1\neq i_2\neq\ldots\neq i_n=1,2,\ldots,n$.

For (i-1), define a left action of G_i on $B\times B$ for each $i=1,2,\ldots,n$,as follows :

$${}^{g}{}_{i}(m,b) = ({}^{g}{}_{i}m, {}^{g}{}_{i}b)$$

for all $m,b\in B$,and $g_i\in G_i$ .This action is continuous for each $i=1,2,\ldots,n$ since the left action of G_i on B is continuous for each $i=1,2,\ldots,n$.

Define a map $\alpha_i:B\times B\to G_i$ for each $i=1,2,\ldots,n$ by $\alpha_i(m,b)=\delta_i(mb)$. Clearly α_i is a homomorphism for each $i=1,2,\ldots,n$. Since B is abelian and δ_i is continuous for each i=1,2,…,n, hence α_i is continuous for each i=1,2,…,n. Now ,we will satisfy the crossed module axioms (CM1) and (CM2).

(CM1) for all $(m,b)\in B\times B$,and $g_i\in G_i$ (i=1,2,…,n);

$$\alpha_i({}^{g_i}(m,b)) = \alpha_i({}^{g_i}m,{}^{g_i}b)$$

$$= \delta_i({}^{g_i}m\,{}^{g_i}b)$$

$$= \delta_i({}^{g_i}(mb)) \quad \text{,(}G_i \text{ acts on the left of B for each i=1,2,…,n)}$$

$$= g_i\,\delta_i(mb)\,g_i^{-1} \quad \text{,(since(}\delta_i,G_i;B\text{) is a left Pro-C crossed module for each i=1,2,…,n)}$$

$$= g_i\,\alpha_i(m,b)\,g_i^{-1}$$

(CM2) for all $(m_1,b_1),(m_2,b_2)\in B\times B$;

$$^{\alpha_i(m_2,b_2)}(m_1,b_1) = {}^{\delta_i(m_2b_2)}(m_1,b_1)$$

$$= ({}^{\delta_i(m_2b_2)}m_1,\ {}^{\delta_i(m_2b_2)}b_1)$$

$$= (m_2b_2m_1b_2^{-1}m_2^{-1},m_2b_2b_1b_2^{-1}m_2^{-1}) \quad \text{,(since (}\delta_i,G_i;B\text{) is a left Pro-C crossed module for each i=1,2,…,n)}$$

$$= (m_2m_1m_2^{-1},b_2b_1b_2^{-1}) \quad \text{,(since B is abelian)}$$

$$=(m_2,b_2)(m_1,b_1)(m_2,b_2)^{-1} .$$

Hence $(\alpha_i,G_i;\ B\times B)$ is a left Pro-C crossed module for each i=1,2,…,n.

Finally,we need only to satisfy (i-2) .For all $(m,b)\in B\times B$, and $g_i\in G_i$ (i=1,2,…,n);

$g_n.(g_{n-1}.(\ldots .(g_2.(g_1.(m,b)))\ldots)$

$$=(g_n.(g_{n-1}.(\ldots.(g_2.(g_1.m))\ldots)\ ,\ g_n.(g_{n-1}.(\ldots .(g_2.(g_1.b))\ldots))$$

$$=(g_{i_n}.(g_{i_{n-1}}.(\ldots.(g_{i_2}.(g_{i_1}.m))\ldots),g_{i_n}.(g_{i_{n-1}}.(\ldots.(g_{i_2}.(g_{i_1}.b))\ldots))$$

$$= g_{i_n}.(\,g_{i_{n-1}}.(\ldots.(\,g_{i_2}.(\,g_{i_1}.\ (m,b)))\ldots)$$

The proof is complete .

To prove (ii) we need to show the following :

(ii-1) $(\alpha_j,G_j;B\times B)$ is a left Pro-C crossed module for each j=1,2,…,i.

(ii-2) $(\eta_l, G_l \times G_l; B \times B)$ is a left Pro-C crossed module for each $l=i+1, i+2, \ldots, n$.

(ii-3) $(g_n,h_n).((g_{n-1},h_{n-1}).(\ldots((g_{i+1},h_{i+1}).(g_i.(\ldots(g_2.(g_1.(m,b)))\ldots)))\ldots))$

$$=(g_{r_n},h_{r_n}).((g_{r_{n-1}},h_{r_{n-1}}).(\ldots((g_{r_{i+1}},h_{r_{i+1}}).(g_{r_i}.(\ldots(g_{r_2}.(g_{r_1}.(m,n)))\ldots)))\ldots))$$

if $r_1 \neq r_2 \neq \ldots \neq r_i = 1,2,\ldots,i$ and $r_{i+1} \neq r_{i+2} \neq \ldots \neq r_n = i+1, i+2, \ldots, n$,where the right side of (ii-3) may take different forms depend on the choice of values of r_j if $r_j=1,2,\ldots,i$ or $r_j=i+1,i+2,\ldots,n$.

The proof of item (ii-1) is similar to (i-1) above .

For (ii-2) , define a left action of $G_l \times G_l$ on $B \times B$ for each $l=i+1, i+2, \ldots, n$ as follows :

$$^{(g_l,h_l)}(m,b) = (^{g_l}m, {}^{h_l}b)$$

for each $(g_l,h_l) \in G_l \times G_l$ and $(m,b) \in B \times B$. This left action is continuous for each $l=i+1, i+2, \ldots, n$, since the action of G_l on B is continuous for each $l=i+1, i+2, \ldots, n$.

Define a map $\eta_l : B \times B \to G_l \times G_l$ by $\eta_l(m,b) = (\delta_l(m), \delta_l(b))$ for each $l=i+1, i+2, \ldots, n$. Clearly η_l is continuous homomorphism , since δ_l is continuous homomorphism for each $l=i+1, i+2, \ldots, n$ .Now, we will satisfy the crossed module axioms (CM1) and (CM2).

(CM1) For all $(m,b) \in B \times B$, and $(g_l,h_l) \in G_l \times G_l$ $(l=i+1, i+2, \ldots, n)$;

$$\eta_l(^{(g_l,h_l)}(m,b)) = \eta_l(^{g_l}m, {}^{h_l}b)$$

$$= (\delta_l(^{g_l}m), \delta_l(^{h_l}b))$$

$$= (g_l\, \delta_l(m)\, g_l^{-1}, h_l\, \delta_l(b)\, h_l^{-1})$$, (since$(\delta_l, G_l; B)$ is a left Pro-C crossed module for each $l=i+1,\ldots,n$)

$$= (g_l,h_l)\, \eta_l(m,b)\, (g_l,h_l)^{-1}$$

(CM2) for all $(m_1,b_1),(m_2,b_2) \in B \times B$;

$$^{\alpha_l(m_2,b_2)}(m_1,b_1) = {}^{(\delta_l(m_2),\delta_l(b_2))}(m_1,b_1)$$

$$= (^{\delta_l(m_2)}m_1, {}^{\delta_l(b_2)}b_1)$$

$$=(m_2m_1m_2^{-1}, b_2b_1b_2^{-1})$$, (since$(\delta_l, G_l; B)$ is a left Pro-C crossed

module for each l=i+1,…,n)

$=(m_2,b_2)(m_1,b_1)(m_2,b_2)^{-1}$.

Therefore $(\eta_l,G_l\times G_l;B\times B)$ is a left Pro-C crossed module for each l=i+1, i+2,…,n.

For(ii-3)

$(g_n,h_n).((g_{n-1},h_{n-1}).(\ldots((g_{i+1},h_{i+1}).(g_i.(\ldots(g_2.(g_1.(m,b)))\ldots)))\ldots))$

$= (g_n.(g_{n-1}.(\ldots(g_{i+1}.(g_i.(\ldots(g_2.(g_1.m))\ldots)))\ldots)),$

$h_n.(h_{n-1}.(\ldots(h_{i+1}.(g_i.(\ldots(g_2.(g_1.b))\ldots)))\ldots)))$

$= (g_n.(g_{n-1}.(\ldots(g_{i+1}.(g_i.(\ldots(g_2.(g_1.m))\ldots)))\ldots)),$

$x_n.(x_{n-1}.(\ldots(x_{i+1}.(x_i.(\ldots(x_2.(x_1.b))\ldots)))\ldots))),$

where $x_j=g_j$, j =1,2,…,i and $x_j=h_j$, j =i+1,i+2,…,n .

$=(g_{r_n}.(g_{r_{n-1}}.(\ldots(g_{r_{i+1}}.(g_{r_i}.(\ldots(g_{r_2}.(g_{r_1}.m))\ldots)))\ldots)),$

$x_{s_n}.(x_{s_{n-1}}.(\ldots(x_{s_{i+1}}.(x_{s_i}.(\ldots(x_{s_2}.(x_{s_1}.b))\ldots)))\ldots))),$

where $r_1\neq r_2\neq\ldots\neq r_i\neq r_{i+1}\neq r_{i+2}\neq\ldots\neq r_n=1,2,\ldots,i\ i+1,i+2,\ldots,n,$

and $s_1\neq s_2\neq\ldots\neq s_i\neq s_{i+1}\neq s_{i+2}\neq\ldots\neq s_n=1,2,\ldots,i\ i+1,i+2,\ldots,n.$

$=(g_{r_n}.(g_{r_{n-1}}.(\ldots(g_{r_{i+1}}.(g_{r_i}.(\ldots(g_{r_2}.(g_{r_1}.m))\ldots)))\ldots)),$

$h_{r_n}.(h_{r_{n-1}}.(\ldots(h_{r_{i+1}}.(g_{r_i}.(\ldots(g_{r_2}.(g_{r_1}.b))\ldots)))\ldots))),$
since the left actions of $G_1,G_2,\ldots, G_n$ on B commute with each other,then;
$x_{s_n}.(x_{s_{n-1}}.(\ldots(x_{s_{i+1}}.(x_{s_i}.(\ldots(x_{s_2}.(x_{s_1}.b))\ldots)))\ldots)))$
$= h_{r_n}.(h_{r_{n-1}}.(\ldots(h_{r_{i+1}}.(g_{r_i}.(\ldots(g_{r_2}.(g_{r_1}.b))\ldots)))\ldots))).$

$=(g_{r_n},h_{r_n}).((g_{r_{n-1}},h_{r_{n-1}}).(\ldots((g_{r_{i+1}},h_{r_{i+1}}).(g_{r_i}.(\ldots(g_{r_2}.(g_{r_1}.(m,b)))\ldots)))\ldots))$

The proof is complete.

Finally , To prove (iii) we need to show the following :

(iii-1) $(\eta_i,G_i\times G_i;B\times B)$ is a left Pro-C crossed module for each i=1,2,…,n.

(iii-2) For each $(m,b)\in B\times B$, and $(g_i,h_i)\in G_i\times G_i$,i=1,2,…,n ;

$(g_n,h_n).((g_{n-1},h_{n-1}).(\ldots((g_2,h_2).((g_1,h_1).(m,b)))\ldots)))\ldots))$

$=(g_{r_n},h_{r_n}).((g_{r_{n-1}},h_{r_{n-1}}).(\ldots((g_{r_2},h_{r_2}).((g_{r_1},h_{r_1}).(m,b)))\ldots)))\ldots))$

where $r_1 \neq r_2 \neq \ldots \neq r_n = 1,2,\ldots,n$.

The proof of (iii-1) is similar to the proof of item (ii-1) above.

For (iii-2),

$$(g_n,h_n).((g_{n-1},h_{n-1}).(\ldots((g_2,h_2).((g_1,h_1).(m,b)))\ldots))\ldots))$$

$$= (g_n.(g_{n-1}.(\ldots(g_2.(g_1.m))\ldots)),\ h_n.(h_{n-1}.(\ldots(h_2.(h_1.b))\ \ldots)))$$

$$=(g_{r_n}.(g_{r_{n-1}}.(\ldots(g_{r_2}.(g_{r_1}.m))\ldots)),h_{r_n}.(h_{r_{n-1}}.(\ldots(h_{r_2}.(h_{r_1}.b))\ldots))),$$

(for $r_1 \neq r_2 \neq \ldots \neq r_n = 1,2,\ldots,n$).

$$=(g_{r_n},h_{r_n}).((g_{r_{n-1}},h_{r_{n-1}}).(\ldots((g_{r_2},h_{r_2}).((g_{r_1},h_{r_1}).(m,b)))\ldots)))\ldots))$$

Hence the proof is complete.

3- Pull-back in the category of left Pro-C n-crossed modules:

Clearly, the left Pro-C n-crossed modules and the continuous n-morphisms between them form a category which we denote by **P_ln-CMod** . Similarly we denote the category of right Pro-C n-crossed modules by **P_rn-CMod** ,and the category of left-right Pro-C n-crossed modules by **P_{lr}n-CMod** .

In this section we will give and study the pull-back in **P_ln-CMod** .

Lemma (3-1):

Let $(\mu_1,\mu_2,..,\mu_n,\mu_{n+1}):(\delta_1,G_1;\delta_2,G_2;..;\delta_n,G_n;B)\to(\sigma_1,K_1;\sigma_2,K_2;..;\sigma_n,K_n;O)$ and, $(\eta_1,\eta_2,..,\eta_n,\eta_{n+1}):(\lambda_1,H_1;\lambda_2,H_2;..;\lambda_n,H_n;M)\to(\sigma_1,K_1;\sigma_2,K_2;..;\sigma_n,K_n;O)$be continuous n-morphisms of left Pro-C n-crossed modules. Then we can find a Pro-C n-crossed module $(\alpha_1,L_1;\alpha_2,L_2;..;\alpha_n,L_n;R)$, and a continuous n-morphisms,

$$(\rho_{G1},\rho_{G2},..,\rho_{Gn},\rho_B):(\alpha_1,L_1;\alpha_2,L_2;..;\alpha_n,L_n;R)\to(\delta_1,G_1;\delta_2,G_2;..;\delta_n,G_n;B)$$

and,

$$(\rho_{H1},\rho_{H2},..,\rho_{Hn},\rho_M):(\alpha_1,L_1;\alpha_2,L_2;..;\alpha_n,L_n;R)\to(\lambda_1,H_1;\lambda_2,H_2;..;\lambda_n,H_n;M)$$

such that,

$$(\mu_1,\mu_2,..,\mu_n,\mu_{n+1})(\rho_{G1},\rho_{G2},..,\rho_{Gn},\rho_B)=(\eta_1,\eta_2,..,\eta_n,\eta_{n+1})(\rho_{H1},\rho_{H2},..,\rho_{Hn},\rho_M)$$

$$\begin{array}{ccc} (\alpha_1,L_1;\alpha_2,L_2;..;\alpha_n,L_n;R) & \xrightarrow{(\rho_{H1},\rho_{H2},..,\rho_{Hn},\rho_M)} & (\lambda_1,H_1;\lambda_2,H_2;..;\lambda_n,H_n;M) \\ \downarrow{\scriptstyle(\rho_{G1},\rho_{G2},..,\rho_{Gn},\rho_B)} & & \downarrow{\scriptstyle(\eta_1,\eta_2,..,\eta_n,\eta_{n+1})} \\ (\delta_1,G_1;\delta_2,G_2;..;\delta_n,G_n;B) & \xrightarrow{(\mu_1,\mu_2,..,\mu_n,\mu_{n+1})} & (\sigma_1,K_1;\sigma_2,K_2;..;\sigma_n,K_n;O) \end{array}$$

Proof:

Let $L_i = \{ (g_i,h_i) \in G_i\times H_i ;\ \mu_i(g_i) = \eta_i(h_i) \}$, for each i = 1 , 2 ,..., n and , $R = \{ (n,m) \in N\times M ;\ \mu_{n+1}(n) = \eta_{n+1}(m) \}$. Clearly, as L_i is closed subgroup of a Pro-C group $G_i\times H_i$,i=1,2,…,n and R is a closed subgroup of a Pro-C group $B\times M$, therefore L_i and R are Pro-C groups for i=1,2,…,n. Define a map $\alpha_i:R\to L_i$ for each i=1,2,…,n , by $\alpha_i(b,m) = (\delta_i(b),\lambda_i(m))$, where $(\delta_i(b),\lambda_i(m)) \in L_i$ for each i=1,2,…,n, since (μ_{n+1},μ_i) and (η_{n+1},η_i) are continuous morphisms of left Pro-C crossed modules for each i=1,2,…,n and $(b,m) \in B\times M$.

Clearly α_i is a continuous homomorphism for each i=1,2,…,n, since each of δ_i and λ_i is a continuous homomorphism for each i=1,2,…,n.

Define $\rho_{Gi}= \pi_1|L_i:L_i\to G_i$ for each i=1,2,…,n to be the restriction of the first projection $\pi_1:G_i\times H_i\to G_i$ on L_i, i=1,2,…,n,and $\rho_B= \pi_1|R:R\to B$ to be the restriction of the first projection $\pi_1:B\times M\to B$ on R. Also define $\rho_{Hi}= \pi_2|L_i:L_i\to H_i$ for each i=1,2,…,n to be the restriction of the second projection $\pi_2:G_i\times H_i\to H_i$ on L_i, i=1,2,…,n, and $\rho_M= \pi_2|R:R\to M$ to be the restriction of the second projection $\pi_2:B\times M\to M$ on R .We need to show the following :

(i) $(\alpha_1,L_1;\alpha_2,L_2;..;\alpha_n,L_n;R)$ is a left Pro-C n-crossed module .

(ii) $(\rho_{G1},\rho_{G2},..,\rho_{Gn},\rho_B)$ and $(\rho_{H1},\rho_{H2},..,\rho_{Hn},\rho_M)$ are continuous n- morphisms such that ,

$$(\mu_1,\mu_2,..,\mu_n,\mu_{n+1})\ (\rho_{G1},\rho_{G2},..,\rho_{Gn},\rho_B) =(\eta_1,\eta_2,..,\eta_n,\eta_{n+1})(\rho_{H1},\rho_{H2},..,\rho_{Hn},\rho_M)$$

To prove (i) we must show the following :

(i-1) $(\alpha_i,L_i;R)$ is a left Pro-C crossed module for each i=1,2,…,n.

(i-2) For each $l_i\in L_i$ and $(b,m)\in R$,

$$(l_n.(l_{n-1}.(\ldots.(l_2.(l_1.(b,m)))\ldots)) = l_{i_n}.(l_{i_{n-1}}.(\ldots.(l_{i_2}.(l_{i_1}.(b,m)))\ldots)) ,$$

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where $i_1 \neq i_2 \neq \ldots \neq i_n = 1,2,\ldots,n$, and $l_i = (g_i,h_i)$, $i=1,2,\ldots,n$.

For (i-1) define a left action of L_i on R by,

$$(g_i,h_i).(b,m) = (g_i.b,h_i.m) \; , \; i=1,2,\ldots,n.$$

for all $(g_i,h_i)\in L_i$ and $(n,m)\in R$. We mention here that $(g_i.b,h_i.m)\in R$ since $(b,m)\in R$ and (μ_{n+1},μ_i) , (η_{n+1},η_i) are continuous morphisms of left Pro-C crossed modules for each $i=1,2,\ldots,n$.

This left action of L_i on R is continuous for each $i=1,2,\ldots,n$,since each of the left action of G_i on B and of H_i on M is continuous for each $i=1,2,\ldots,n$. Now ,we need only to satisfy the crossed module axioms (CM1) and (CM2) .

(CM1) For all $(b,m)\in R$ and $(g_i,h_i)\in L_i$, $(i=1,2,\ldots,n)$;

$$\begin{aligned}\alpha_i({}^{(g_i,h_i)}(b,m)) &= \alpha_i({}^{g_i}b,{}^{h_i}m)\\ &= (\delta_i({}^{g_i}b),\, \lambda_i({}^{h_i}m))\\ &= (g_i\, \delta_i(b)\, g_i^{-1},\, h_i\, \lambda_i(m)\, h_i^{-1})\\ &= (g_i,h_i)\, (\delta_i(b),\lambda_i(m))\, (g_i,h_i)^{-1}\\ &= (g_i,h_i)\, \alpha_i(b,m)\, (g_i,h_i)^{-1}\end{aligned}$$

(CM2) for all $(b_1,m_1),(b_2,m_2)\in R$;

$$\begin{aligned}{}^{\alpha_i(b_2,m_2)}(b_1,m_1) &= {}^{(\delta_i(b_2),\, \lambda_i(m_2))}(b_1,m_1)\\ &= ({}^{\delta_i(b_2)}b_1,\, {}^{\lambda_i(m_2)}m_1)\\ &=(b_2b_1b_2^{-1},m_2m_1m_2^{-1})\\ &=(b_2,m_2)(b_1,m_1)(b_2,m_2)^{-1}\; .\end{aligned}$$

For (i-2) Let $(b,m)\in R$, and $(g_i,h_i)\in L_i$,$i=1,2,\ldots,n$, then ;

$$\begin{aligned}&(g_n,h_n).((g_{n-1},h_{n-1}).(\ldots.((g_2,h_2).((g_1,h_1).(b,m)))\ldots)))\ldots))\\ &= (g_n.(g_{n-1}.(\ldots.(g_2.(g_1.b))\ldots)),\, h_n.(h_{n-1}.(\ldots.(h_2.(h_1.m))\ldots)))\\ &=(g_{i_n}.(g_{i_{n-1}}.(\ldots.(g_{i_2}.(g_{i_1}.b))\ldots)),h_{i_n}.(h_{i_{n-1}}.(\ldots.(h_{i_2}.(h_{i_1}.m))\ldots))),\\ &\qquad\qquad (\text{for } i_1\neq i_2\neq\ldots\neq i_n=1,2,\ldots,n).\\ &=(g_{i_n},h_{i_n}).((g_{i_{n-1}},h_{i_{n-1}}).(\ldots.((g_{i_2},h_{i_2}).((g_{i_1},h_{i_1}).(b,m)))\ldots))\end{aligned}$$

Therefore, $(\alpha_1,L_1;\alpha_2,L_2;..;\alpha_n,L_n;R)$ is a left Pro-C n-crossed module .

For (ii), we will first show that,

$$(\rho_{G1},\rho_{G2},..,\rho_{Gn},\rho_B):(\alpha_1,L_1;\alpha_2,L_2;..;\alpha_n,L_n;R)\to(\delta_1,G_1;\delta_2,G_2;..;\delta_n,G_n;B)$$

is a continuous n-morphism. To do this we need to show the following:

(ii-1) $(\rho_{Gi},\rho_B):(\alpha_i,L_i;R)\to(\delta_i,G_i;B)$ is a continuous morphism of left Pro-C crossed modules for each i=1,2,…,n ,i.e. we want to show that $\delta_i\rho_B = \rho_{Gi}\alpha_i$ and

$$\rho_B({}^{(g_i,h_i)}(b,m)) = {}^{\rho_{Gi}(g_i,h_i)}\rho_B(b,m)$$

for all $(b,m)\in R$ and $(g_i,h_i)\in L_i$,(i=1,2,…,n).

For all $(b,m)\in R$, we have

$$\delta_i\rho_B(b,m)= \delta_i(b)= \rho_{Gi}(\delta_i(b),\lambda_i(m))= \rho_{Gi}\alpha_i(b,m).$$

Therefore $\delta_i\rho_B = \rho_{Gi}\alpha_i$ for each i=1,2,…,n.

For each $(b,m)\in R$ and $(g_i,h_i)\in L_i$, we have:

$$\rho_B({}^{(g_i,h_i)}(b,m)) = \rho_B({}^{g_i}b,{}^{h_i}m) = {}^{g_i}b = {}^{\rho_{Gi}(g_i,h_i)}\rho_B(b,m).$$

Similarly, we can prove that ,

$$(\rho_{H1},\rho_{H2},..,\rho_{Hn},\rho_M) : (\alpha_1,L_1;\alpha_2,L_2;..;\alpha_n,L_n;R)\to(\lambda_1,H_1;\lambda_2,H_2;..;\lambda_n,H_n;M)$$

is a continuous n-morphism of left Pro-C n-crossed modules.

Finally , we need only to show that,

$$(\mu_1,\mu_2,..,\mu_n,\mu_{n+1})(\rho_{G1},\rho_{G2},..,\rho_{Gn},\rho_B) =(\eta_1,\eta_2,..,\eta_n,\eta_{n+1})(\rho_{H1},\rho_{H2},..,\rho_{Hn},\rho_M)$$

and this is clear, since one can easly show that $\mu_i\,\rho_{Gi}=\eta_i\,\rho_{Hi}$ for each i=1,2,… n, and $\mu_{n+1}\rho_B=\eta_{n+1}\rho_M$.

We mention here that the left Pro-C n-crossed module $(\alpha_1,L_1;\alpha_2,L_2;..;\alpha_n,L_n;R)$ and the continuous n-morphisms $\rho_G=(\rho_{G1},\rho_{G2},..,\rho_{Gn},\rho_B)$and $\rho_H=(\rho_{H1},\rho_{H2},..,\rho_{Hn},\rho_M)$ as constructed in the proof of lemma (3-1) are represent the **pull-back** of $(\mu_1,\mu_2,..,\mu_n,\mu_{n+1})$and $(\eta_1,\eta_2,..,\eta_n,\eta_{n+1})$ in **P_ln-Cmod** , which we denote by $((\alpha_1,L_1;\alpha_2,L_2;..;\alpha_n,L_n;R);\rho_G;\rho_H)$.

In the followig theorem we will give the universal property of the pull-back $((\alpha_1,L_1;\alpha_2,L_2;..;\alpha_n,L_n;R);\rho_G;\rho_H)$.

Theorem (3-2):

Let $((\alpha_1,L_1;\alpha_2,L_2;..;\alpha_n,L_n;R);\rho_G;\rho_H)$ be the bull-back of $(\mu_1,\mu_2,..,\mu_n,\mu_{n+1})$ and $(\eta_1,\eta_2,..,\eta_n,\eta_{n+1})$ in **P$_l$n-CMod** as constructed in the proof of lemma (3-1). If

$$(\theta_1,\theta_2,..,\theta_n,\theta_{n+1}):(\beta_1,A_1;\beta_2,A_2;..;\beta_n,A_n;T)\rightarrow(\delta_1,G_1;\delta_2,G_2;..;\delta_n,G_n;B)$$

and ,

$$(\varphi_1,\varphi_2,..,\varphi_n,\varphi_{n+1}):(\beta_1,A_1;\beta_2,A_2;..;\beta_n,A_n;T)\rightarrow(\lambda_1,H_1;\lambda_2,H_2;..;\lambda_n,H_n;M)$$

are continuous n-morphisms of left Pro-C n-crossed modules such that,

$$(\mu_1,\mu_2,..,\mu_n,\mu_{n+1})(\theta_1,\theta_2,..,\theta_n,\theta_{n+1})=(\eta_1,\eta_2,..,\eta_n,\eta_{n+1})(\varphi_1,\varphi_2,..,\varphi_n,\varphi_{n+1})$$

then the n-morphism,

$$(\xi_1,\xi_2,..,\xi_n,\xi_{n+1}):(\beta_1,A_1;\beta_2,A_2;..;\beta_n,A_n;T)\rightarrow(\alpha_1,L_1;\alpha_2,L_2;..;\alpha_n,L_n;R)$$

which is defined by the group homomorphisms, $\xi_1(a_1)=(\theta_1(a_1),\varphi_1(a_1))$, $\xi_2(a_2)=(\theta_2(a_2),\varphi_2(a_2)),\ldots\ldots,\xi_n(a_n)=(\theta_n(a_n),\varphi_n(a_n))$, $\xi_{n+1}(t)=(\theta_{n+1}(t),\varphi_{n+1}(t))$,is the unique continuous n-morphism of left Pro-C n-crossed modules satisfying:

$(\rho_{G1},\rho_{G2},..,\rho_{Gn},\rho_B)(\xi_1,\xi_2,..,\xi_n,\xi_{n+1})=(\theta_1,\theta_2,..,\theta_n,\theta_{n+1})$, and;

$(\rho_{H1},\rho_{H2},..,\rho_{Hn},\rho_M)(\xi_1,\xi_2,..,\xi_n,\xi_{n+1})=(\varphi_1,\varphi_2,..,\varphi_n,\varphi_{n+1})$.

Proof:

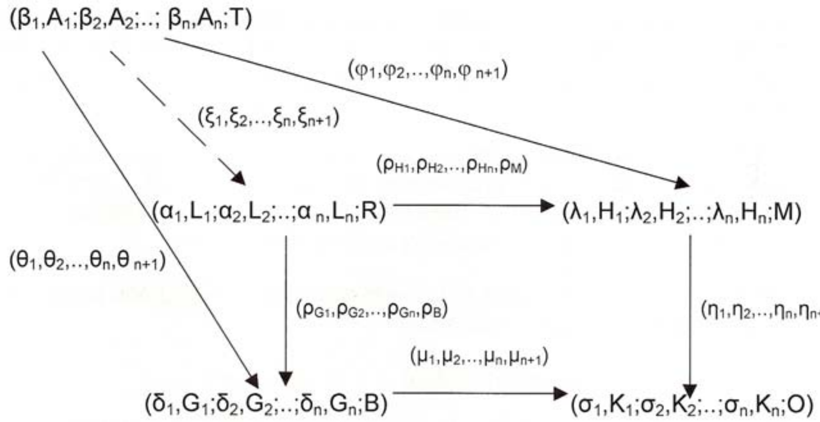
We have to prove the following :

(i) $(\xi_1,\xi_2,..,\xi_n,\xi_{n+1})$is a continuous n-morphism of left Pro-C n-crossed modules.

(ii) $(\rho_{G1},\rho_{G2},..,\rho_{Gn},\rho_B)(\xi_1,\xi_2,..,\xi_n,\xi_{n+1})=(\theta_1,\theta_2,..,\theta_n,\theta_{n+1})$,and,

$(\rho_{H1},\rho_{H2},..,\rho_{Hn},\rho_M)(\xi_1,\xi_2,..,\xi_n,\xi_{n+1})=(\varphi_1,\varphi_2,..,\varphi_n,\varphi_{n+1})$.

(iii) Uniqueness of $(\xi_1,\xi_2,..,\xi_n,\xi_{n+1})$.



For (i),we need to show that $(\xi_i,\xi_{n+1}):(\beta_i,A_i;T)\rightarrow(\alpha_i,L_i;R)$ be a continuous morphism of left Pro-C crossed modules for each i=1,2,…,n , i.e. want $\alpha_i\xi_{n+1}=\xi_i\beta_i$ and $\xi_{n+1}({}^{a_i}t) = {}^{\xi_i(a_i)}\xi_{n+1}(t)$,(i=1,2,…,n) ,for all $t\in T$ and $a_i\in A_i$.

For any $t\in T$, we have,

$$\alpha_i\xi_{n+1}(t)= \alpha_i(\theta_{n+1}(t),\varphi_{n+1}(t))$$

$$= (\delta_i\theta_{n+1}(t),\lambda_i\varphi_{n+1}(t))$$

$$= (\theta_i\beta_i(t),\varphi_i\beta_i(t))$$,(since (θ_i,θ_{n+1}) and $(\varphi_i,\varphi_{n+1})$ are continuous morphism of left Pro-C crossed modules for each i=1,2,…,n)

$$=\xi_i\beta_i(t) .$$

Therefore $\alpha_i\xi_{n+1}=\xi_i\beta_i$,i=1,2,…,n.

Now, for any $t\in T$ and $a_i\in A_i$ (i=1,2,…,n),we have :

$$\xi_{n+1}({}^{a_i}t) =(\theta_{n+1}({}^{a_i}t),\varphi_{n+1}({}^{a_i}t))$$

$$=({}^{\theta_i(a_i)}\theta_{n+1}(t),{}^{\varphi_i(a_i)}\varphi_{n+1}(t))$$,(since (θ_i,θ_{n+1}) and $(\varphi_i,\varphi_{n+1})$ are continuous morphism of left Pro-C crossed modules for each i=1,2,…,n)

$$= {}^{(\theta_i(a_i),\varphi_i(a_i))}(\theta_{n+1}(t),\varphi_{n+1}(t))$$

$$= {}^{\xi_i(a_i)}\xi_{n+1}(t) .$$

Since θ_i and φ_i are continuous morphisms for each i=1,2,…,n+1,then ξ_i is a continuous morphism for each i=1,2,…,n+1. Hence the n- morphism $(\xi_1,\xi_2,..,\xi_n,\xi_{n+1})$ is continuous.From the definitions of ξ_i, i=1,2,…,n+1, one can easly deduce (ii).

Finally, we want to prove the uniqueness of $(\xi_1,\xi_2,..,\xi_n,\xi_{n+1})$ which is equivalent to prove the uniqueness of each of ξ_i, i=1,2,…,n+1.To prove the uniqueness of $\xi_i:A_i\to L_i$,suppose that there is another continuous homomorphism $\psi_i:A_i\to L_i$,satisfying $\rho_{Gi}\,\psi_i=\theta_i$ and $\rho_{Hi}\,\psi_i=\varphi_i$.

Since $\rho_{Gi}=\pi_1|L_i:L_i\to G_i$, $\rho_{Hi}=\pi_2|L_i:L_i\to H_i$ and $\rho_{Gi}\psi_i(a_i)=\theta_i(a_i)$ and $\rho_{Hi}\psi_i(a_i)=\varphi_i(a_i)$ for all $a_i\in A_i$, then ψ_i must be defined as ;

$$\psi_i(a_i)=(\theta_i(a_i),\varphi_i(a_i))$$

Hence $\psi_i=\xi_i$ for each i=1,2,…,n. therefore ξ_i is unique for each i=1,2,…,n. Similarly we can prove the uniqueness of ξ_{n+1} . Hence $(\xi_1,\xi_2,..,\xi_n,\xi_{n+1})$ is unique .

This complete the proof .

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